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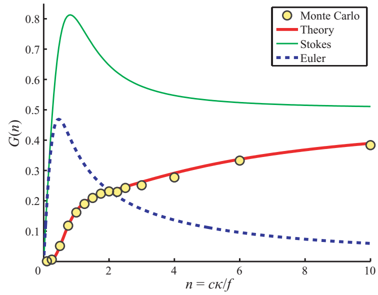
Focus on Fluids

Could waves mix the ocean?



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A finite-amplitude propagating wave induces a drift in fluids. Understanding how drifts produced by many waves disperse pollutants has broad implications for geophysics and engineering. Previously, the effective diffusivity was calculated for a random set of small-amplitude surface and internal waves. Now, this is extended by Bühler & Holmes-Cerfon (*J. Fluid Mech.*, 2009, this issue, vol. 638, pp. 5–26) to waves in a rotating shallow-water system in which the Coriolis force is accounted for, a necessary step towards oceanographic applications. It is shown that interactions of finite-amplitude waves affect particle velocity in subtle ways. An expression describing the particle diffusivity as a function of scale is derived, showing that the diffusivity can be substantially reduced by rotation.

Keywords. Waves, drift, diffusion, nonlinearity

1. Introduction

A slick of pollutants released at a depth of a few hundred metres in the ocean diffuses horizontally for a couple of kilometres per month (see e.g. Ledwell, Watson & Law 1993). Molecular diffusion acting alone would make that happen in about a hundred million years, just as it would take years for the aroma of coffee to diffuse from the cup to the nose. A usual suspect – convective fluid flows – must be responsible. The coffee aroma is principally carried by thermal convection in interior air, but the ocean is stably stratified with density (owing to temperature and salinity differences) increasing with depth. Could the random flows that cause diffusion be produced by waves which are more ubiquitous than currents, particularly deep in the ocean?

The flow associated with a plane monochromatic wave with amplitude v_k, wavelength $2\pi/k$ (k being the wavenumber) and frequency ω_k looks very simple: the velocity v depends on the coordinate x and time t sinusoidally, $v(x,t)=v_k\sin(kx-\omega_k t)$. This simplicity is deceptive, for example, how far does a particular fluid particle move during a time t? To answer, one needs to use the 'Lagrangian' description, defined in terms of the current particle coordinate $x(t)$ whose time derivative is the velocity:

$$\dot{x}(t) = v_k \sin[kx(t) - \omega_k t]. \tag{1.1}$$

This is a nonlinear equation for $x(t)$, even for a linear plane monochromatic wave. Moreover, its solution is not periodic. Assuming that the wave amplitude is small, $v_k \ll \omega_k/k$, one can solve the equation iteratively, $x(t)=x_0+x_1(t)+x_2(t)\ldots$. This procedure yields a periodic oscillation at the first order in amplitude, $x_1=(v_k/\omega_k)\cos(kx_0-\omega_k t)$, and the 'Stokes drift' at second order, $x_2(t)=kv_k^2 t/2\omega_k+(kv_k^2/2\omega_k^2)\sin 2(kx_0-\omega_k t)$. At first order in the amplitude v_k the

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perturbation propagates, while at second order the fluid itself flows with mean velocity $u_k = kv_k^2/2\omega_k$.

Even in this simple one-dimensional example the Stokes drift at second order in wave amplitude is only half of the story because nonlinear corrections to the linear wave field (1.1) stemming from the full set of the dynamical equations (including boundary conditions) may arise at the same order in wave amplitude. These nonlinear corrections often contain a systematic mean-flow response that can compete with the Stokes drift at second order. Thus both the Stokes drift and the mean-flow response must be computed to obtain the mean velocity at leading order.

The situation is substantially more interesting when there are many waves. The mean Stokes drift velocity is just the integral over the wave spectrum $\langle \boldsymbol{u} \rangle = \int \boldsymbol{u_k}\, \mathrm{d}\boldsymbol{k} = \int \boldsymbol{k}(|v_{\boldsymbol{k}}|^2/2\omega_k)\, d\boldsymbol{k}$. The brackets denote the average over a time that is large compared with all the wave periods; on a shorter time scale, the drift velocity fluctuates, and the variance of such fluctuations determines the dispersion of the fluid particles. To illustrate this, consider a situation in which either the Stokes drift is zero or we are in a reference frame moving at that velocity $\langle \boldsymbol{u} \rangle$. Then we are interested in the mean squared displacement, which is given by the usual diffusion expression:

$$\langle x^2(t) \rangle = \int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t'' \langle \boldsymbol{u}(t') \cdot \boldsymbol{u}(t'') \rangle = 2t \int_0^t \langle \boldsymbol{u}(0) \cdot \boldsymbol{u}(t') \rangle \, \mathrm{d}t' = 2Dt. \tag{1.2}$$

The diffusivity D is thus proportional to the fourth power of the wave amplitudes v_k.

Herterich & Hasselman (1982) were the first to calculate such diffusivity for surface waves, followed by Sanderson & Okubo (1988), who did it for horizontally isotropic internal waves. Balk & McLaughlin (1999) derived the diffusivity for a general dispersion relation for the one-dimensional case using weak turbulence theory as described in Zakharov, Lvov & Falkovich (1992). However, geophysically relevant applications (e.g. large-scale oceanic diffusion) require that the effect of fluid rotation on the waves is taken into account. Also, as noted above, nonlinear corrections to the wave description can induce a mean flow which must be considered in concert with the Stokes drift. Bühler & Holmes-Cerfon (2009, this issue, vol. 638, pp. 5–26) address these two key topics, by calculating self-consistently the effective diffusivity for a random set of waves that have small yet finite amplitude in a rotating shallow-water equations system.

2. Overview

The shallow-water equations describe the evolution of the two-dimensional (horizontal) velocity and the fluid depth. Importantly, shallow-water equations account for both the effect of gravity and rotation through the action of the Coriolis force. For a small velocity and an almost-constant depth, the linearized equations describe non-interacting inertia–gravity waves with the dispersion relation $\omega_\kappa^2 = f^2 + c^2\kappa^2$, where f is the Coriolis parameter and is equal to twice the rotation frequency of the system, $c^2 = gh$, the product of the gravitational acceleration and the fluid depth, and κ is the magnitude of the wavenumber vector. The frequency is bounded from zero; i.e. f is a cutoff frequency, which sets a natural averaging time.

The central point of Bühler & Holmes-Cerfon (2009) is to calculate in a self-consistent manner the two-time correlation function of the Lagrangian velocities which enters (1.2). That requires first solving the full nonlinear shallow-water equations (written in the fixed, Eulerian reference frame) up to the second order in the small parameter $\kappa v_\kappa/\omega_\kappa$. That inherently nonlinear second-order contribution to the mean

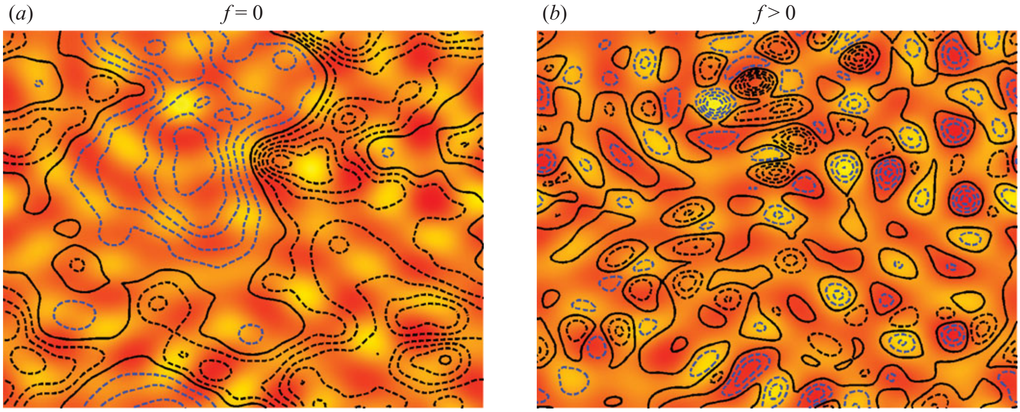

FIGURE 1. Snapshots of random surface height field (red/yellow shading) and contours of the second-order Lagrangian streamfunction (black dashed = positive values, blue dashed = negative values, black solid = zero contour), viewed in the x–y plane.

flow must be taken together with the Stokes drift calculated from the first-order linear wave-like contribution to calculate the diffusivity (neglecting one or the other contribution marred some previous works on the subject).

Calculating the resulting fourth-order expression for the diffusivity is straightforward but technically demanding. It is significantly simplified by assuming that the wave amplitudes v_κ are Gaussian random variables with a zero mean and variance $E(\kappa) = \langle |v_\kappa|^2 \rangle$. For sufficiently wide distribution of small-amplitude waves in κ-space, this is a reasonable approximation, since phases of different waves can be considered to be random, which leads to Gaussianity (see e.g. Zakharov *et al.* 1992). For a Gaussian random field, a fourth-order correlation function is broken into a product of two second-order correlation functions. The resulting general formula from Bühler & Holmes-Cerfon (2009) (derived explicitly for inertia–gravity waves) expresses the diffusivity D as the squared wave energy spectrum $S(\kappa)^2$ integrated with the non-negative spectral diffusivity density $G(\kappa)$, i.e.

$$2\mathrm{D} = \frac{1}{2\pi c^3} \frac{1}{2} \int G(\kappa) S(\kappa)^2 \, \mathrm{d}\kappa. \tag{2.1}$$

In the figure beside the title, the analytical formula for G as a function of non-dimensional wavenumber $n = \kappa c/f$ is plotted, along with the diffusivity densities induced by the Stokes drift (solid green line) and Eulerian nonlinear flow (dashed blue line), and it is clear that the correct diffusivity is due to their joint action. The authors also checked their calculations using Monte Carlo simulations of the velocity field.

It is clear that rotation reduces the effective diffusivity, a key finding of the paper of Bühler & Holmes-Cerfon (2009). In figure 1, snapshots of a random surface height field and the second-order Lagrangian streamfunction are plotted from specific Monte Carlo simulations of Bühler & Holmes-Cerfon (2009). These were calculated from the same realization of a Gaussian random wave field, so that the height fields are comparable. In figure 1(a), the streamfunction is calculated with rotation set to zero, while in figure 1(b) the rotation is very strong. The height fields are virtually identical (after being rescaled to have the same scale), but the streamlines show drastically different patterns: the rotating streamlines are closely aligned with the height contours, while the non-rotating streamlines show a much larger-scale pattern. This hints at the much stronger diffusive efficacy of the non-rotating streamfunction.

Alternatively, remember that increasing rotation increases wave frequency for a given wavenumber and thus decreases the Stokes drift (assuming that v_κ is constant). That makes it natural to expect that the effective diffusivity is a decreasing function of the Coriolis parameter f. The explicit formula presented in the paper quantifies this decrease and reveals an interesting cancellation between the Lagrangian and the second-order Eulerian contributions at the root of this decrease. In the limit of strong rotation the diffusivity is shown to decay as the fifth power of f while the Stokes drift contribution decays as the first power, showing yet again that an analysis in terms of the Stokes drift alone can be very misleading.

3. Future

One can now use the formula derived by Bühler & Holmes-Cerfon (2009) to analyse oceanographic data on a large-scale pollutant dispersion, to try to answer the question originally posed: could waves mix the ocean? The theory itself can be developed in different directions. First, one can go beyond the shallow-water equations model, taking into account the full three-dimensional structure of the wave flow. Second, one can go to higher wave amplitudes, considering mixing by finite-amplitude and breaking waves. An example of other interesting finite-amplitude effects is described in the work of Balk (2006), where diffusion appears already in the second order in wave amplitude (for an anisotropic distribution of compressible wave flows). A third promising direction is the consideration of the joint action of waves and vortical currents (see e.g. Polzin & Ferrari 2004; Vucelja, Falkovich & Fouxon 2007) which are, of course, ubiquitous in the oceans. Through such generalizations, it can be hoped that the ideas of the paper of Bühler & Holmes-Cerfon (2009) can actually be applied to real problems of dispersion and diffusion in the oceans.

References

Balk, A. M. 2006 Wave turbulent diffusion due to the Doppler shift. *J. Stat. Mech.* P08018.

Balk, A. M. & McLaughlin, R. M. 1999 Passive scalar in a random wave field: the weak turbulence approach. *Phys. Lett.* A **256** (4), 299–306.

Bühler, O. & Holmes-Cerfon, M. 2009 Particle dispersion by random waves in rotating shallow water. *J. Fluid Mech.* **638**, 5–26.

Herterich, K. & Hasselmann, K. 1982 The horizontal diffusion of tracers by surface waves. *J. Phys. Oceanogr.* **12**, 704–712.

Ledwell, J., Watson, A. & Law, C. 1993 Evidence for slow mixing across the pycnocline from an open-ocean tracer-release experiment. *Nature* **364**, 701–703.

Polzin, K. & Ferrari, R. 2004 Isopycnal dispersion in natre. *J. Phys. Oceanogr.* **34**, 247–257.

Sanderson, B. G. & Okubo, A. 1988 Diffusion by internal waves. *J. Geophys. Res.* **93**, 3570–3582.

Vucelja, M., Falkovich, G. & Fouxon, I. 2007 Clustering of matter in waves and currents. *Phys. Rev.* E **75**, 065301.

Zakharov, V., Lvov, V. & Falkovich, G. 1992 *Kolmogorov Spectra of Turbulence*. Springer.